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Run Time Statistics in Models of Growth in Disordered Media

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Models for growth in disordered media, which are deterministic once the random environment is specified, can be transformed into stochastic growth processes with a time-dependent noise. The mapping generally induces a memory in the stochastic process. This transformation is proposed and discussed in detail. An approximation scheme is worked out analytically and possible applications are discussed.

KEY WORDS: Irreversible growth processes; quenched disorder; self-organized criticality; invasion percolation; conditional probability.

1. INTRODUCTION

Off-equilibrium dynamics and the effect of quenched disorder are two of the most interesting topics of research in statistical mechanics. When studying systems that involve dynamics in a random medium, which possesses both of these ingredients, one faces a lack of powerful theoretical methods and is often forced to resort to numerical simulation. On the other hand, the dynamics of disordered systems is characterized by a large variety of exotic phenomena, such as stretched exponential relaxation and memory effects in charge density waves,[1] aging in spin glasses,[2] self-organized criticality,[3] and temporal multiscaling.[4] The intention of this paper is to suggest a method that can contribute both to an advance in the theoretical understanding and to a broadening of the amount of information that can be extracted via numerical simulation.

We will focus on a model for the evolution of an aggregate where the mechanism of growth depends only on the extreme statistics of the random

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environment. In other words, randomness enters the model through a random potential $\varepsilon(x)$ that depends on the position x; growth occurs at the point that, among those accessible for growth, has the smallest random potential. This mechanism is a model for various situations, such as fluid flow in porous media or zero-temperature dynamics in statistical systems with disorder. Classic examples of models of this kind are invasion percolation.[5] and the kinetic random-field Ising model.[6] Models recently introduced for interface growth in random media[4] and of biological evolution[3] are based on this same mechanism. The great interest in these models, apart from their applications, lies in the fact that they show long-range space-time correlations typical of a statistical system at a second-order critical point. This happens "spontaneously" without the *temperature fine tuning* that makes a system like the Ising model critical. For this reason these models are typical examples of the very active field of self-organized criticality.

This paper is based on the idea that such models may be viewed as cognitive processes that involve testing the random environment. This observation can be translated into a mathematical formalism, using the rules of conditional probability, that specifies the evolution of the *run time statistics* of the random environment. This is the *effective* probability distribution in which the information on the history of the process is stored in a conditional way. This idea was introduced by Pietronero and Schneider in the fixed scale transformation[7] approach to invasion percolation.[8] It has, in our opinion, a much wider application and provides a general mapping of a quenched growth process, which is deterministic with a space-dependent randomness, to an annealed stochastic process, which is a process with a time-dependent randomness. The latter is specified by a probability distribution for the individual growth events at each time that contains the average on all realizations of the disorder with that given history. The growth probability distribution embodies the *screening* effect which is the origin of the interesting behavior of these systems. It also provides the probability of a given realization and allows one in principle to perform averages on the space of all realizations. In the following exposition we intend to give more emphasis to the application of these ideas than to the specific results for some particular model. Along with some references to particular models discussed in the paper, the concluding section is an attempt to remedy this choice of generality.

The paper is organized as follows: the model we deal with is defined in Section 2, where also the notations are introduced. In Section 3 the mapping of the quenched growth process to a stochastic process is discussed in general, introducing the concept of the run time statistics. This is the first important result. In the following section the basic equations are then

solved in a simple approximation for the empirical distribution that in its turn provides information on all the other quantities that characterize the process. The case of invasion percolation is discussed first to illustrate the method, the approximations involved, and how these can be controlled. The solution of the general case is an easy extension of the results on invasion percolation. A general relation for the critical threshold that marks the location of the discontinuity in the empirical distribution is derived. This extends a known rigorous result for invasion percolation[9] to the general model. A particular limit of the model that is realized in recently proposed models for interface growth[4] and biological evolution[3] is next analyzed in detail. There the relaxation to the stationary state is investigated, providing an explicit expression for the relaxation time. The following section deals with the asymptotic limit of an infinite cluster. The characterization of the process in this limit allows the evaluation of the avalanche size distribution. The relevance of the present approach to applications of methods like the dynamic renormalization group and the fixed scale transformation is discussed. Other applications and extensions of the ideas and results contained in the previous sections are discussed in the last section.

2. THE MODEL

We consider a model defined on a lattice. A random variable $\varepsilon_i \in [0, 1]$ is assigned to every site of the lattice. These are all independent and they are drawn from the same distribution $P_{0,0}(x) = \text{Prob}(\varepsilon_i < x)$. Here and in the following we will use lowercase letters for probability densities, while uppercase letters stand for the corresponding distribution [e.g., $F(x) = \int_0^x f(y)\, dy$]. Both ε_i and the their distributions can be defined in $[0, 1]$ without loss of generality. We will refer to site models, the extension to the bond version being trivial. Once the disorder is assigned, the process is deterministic. This is best explained by the example of site invasion percolation[5,9] (IP), a model for fluid displacement in porous media. One seed site i_0 is initially chosen. Among its nearest neighbor, the one with the smallest random variable i_1 is added to i_0 at the $t = 1$ time step. Next the site with the smallest random variable among the perimeter sites of the two-site cluster $\mathscr{C}_1 \equiv \{i_0, i_1\}$ is selected and added to the cluster. At a generic time t the next site to be added to $\mathscr{C}_t = \{i_0, \ldots, i_{t-1}\}$ is the one with the smallest ε_i, where i labels one of the sites on the perimeter $\partial\mathscr{C}_t$ of the cluster $\mathscr{C}_t$.

This model is generalized as follows: growth starts from a set of seed sites $\mathscr{C}_0$. This defines also a corresponding set of active sites $\partial\mathscr{C}_0$, usually on the perimeter of $\mathscr{C}_0$. Among these the one with the smallest random variable, i.e., $\varepsilon_{i_0} = \min\{\varepsilon_j;\, j \in \partial\mathscr{C}_0\}$, is selected as the initiator of the growth

event at time 0. A deterministic rule then specifies the set of sites $\mathscr{G}_0$ that, along with i_0, become part of the new cluster $\mathscr{C}_1 = \mathscr{C}_0 \cup \mathscr{G}_0 \cup \{i_0\}$. In the IP case $\mathscr{G}_0 \equiv \varnothing$. The same mechanism is repeated at later times: given the cluster set $\mathscr{C}_t$, the set of active sites $\partial\mathscr{C}_t$ is identified. Among these the one, i_t, with the smallest variable initiates the growth event by which a set $\mathscr{G}_t$, together with i_t, is removed from $\partial\mathscr{C}_t$ and is added to the cluster $\mathscr{C}_{t+1} = \mathscr{C}_t \cup \mathscr{G}_t \cup \{i_t\}$.

In Section 4.3 we discuss some recently proposed models for interface growth[4] in a disordered medium that fit this general definition of the model. Here along with i_t also sites from the close neighborhood of i_t are allowed to grow to mimic the effect of surface tension. Also a recent model of biological evolution[3] exploits the same mechanism.

This class of models are simple prototypes of systems that display a self-organized critical behavior. Indeed IP is known[5,9] to reproduce the geometrical properties of standard percolation right at criticality without fine tuning any temperature-like parameter.

3. RUN TIME STATISTICS

The central idea, originally introduced by Pietronero and Schneider[8] in the fixed scale transformation approach to IP, of the present approach is that, as the process goes on, it accumulates information on the random environment. Using the rules of conditional probability, this information can be expressed in the effective distributions of the random variables in $\partial\mathscr{C}_t$. At any time step, all the variables ε_i, for $i \in \partial\mathscr{C}_t$, are tested to decide which is the minimum, ε_{i_t}. For the one ($i = i_t$) that "wins" we acquire the information that it has been the smallest among all. For the others we know that they were not the smallest at that time. This can be directly translated into a formula for the effective probability density $\rho_i(x, t)$ of the variables ε_i at time t, which we will call hereafter run time statistics (RTS). Before doing this it is essential to note that the distribution of ε_i for $i \in \partial\mathscr{C}_t$ depends only on the number τ_i of times it has been tested or, which is the same, on the time τ_i it has been in the set of active sites. We then choose the notation $p_{\tau,t}(x)$ for the RTS of variables that have been tested τ times and $n_{\tau,t}$ for the number of these sites. Here $N_t = \sum_\tau n_{\tau,t} = |\partial\mathscr{C}_t|$ is the total number of sites in $\partial\mathscr{C}_t$. Once the distribution $n_{\tau,t}$ and the RTS $p_{\tau,t}(x)$ are given, the probability that a variable ε_i with $\tau_i = \tau$ initiates growth is

$$\mu_{\tau,t} = \int_0^1 p_{\tau,t}(x) \prod_k [1 - P_{k,t}(y)\, dy]^{n_{k,t} - \delta_{k,\tau}} \tag{1}$$

This is indeed the probability that $\varepsilon_j > \varepsilon_i$ for all $j \neq i$. Here the Kronecker delta $\delta_{k,\tau}$ comes because the contribution of the variable ε_i itself does not

appear in the product. The normalization of this growth probability distribution (GPD) is easily proved introducing the function

$$Z_t(x) = \prod_\tau [1 - P_{\tau,t}(x)]^{n_{\tau,t}} \tag{2}$$

and observing that $\sum_\tau n_{\tau,t}\mu_{\tau,t} = -\int_0^1 \partial_x Z_t(x)\, dx = 1$.

Once a site i_t with $\tau_{i_t} = \tau$ is selected according to this GPD, its density is updated in a conditional way with the information that $\varepsilon_{i_t} < \varepsilon_j$ for all $j \neq i_t$. So it becomes

$$m_t(x \mid \tau) = \frac{1}{\mu_{\tau,t}}\, p_{\tau,t}(x) \prod_k [1 - P_{k,t}(x)]^{n_{k,t} - \delta_{k,\tau}} \tag{3}$$

The distribution of the other ε_j for $j \in \partial\mathscr{C}_t$ is updated with the information that $\varepsilon_j > \varepsilon_{i_t}$. The rules of conditional probability, if $\tau_j = \theta$, yield

$$p_{\theta+1,t+1}(x) = \frac{M_t(x \mid \tau)}{\int_0^1 p_{\theta,t}(y)\, M_t(y \mid \tau)\, dy}\, p_{\theta,t}(x) \tag{4}$$

which also expresses the fact that now the variable ε_j has been tested $\tau_j + 1$ times. The sites in $\mathscr{G}_t$ are added to the cluster. Their RTS no longer evolve. The other ones remain in the set $\partial\mathscr{C}_{t+1}$ of active sites. Finally $n_{0,t+1}$ sites eventually enter the interface $\partial\mathscr{C}_{t+1}$ as they are reached by the growing cluster. Their probability distribution is of course the original one $p_{0,t+1}(x) = p_{0,0}(x)$.

Accordingly the distribution $n_{\theta,t}$ is updated: $n_{\theta+1,t+1} = n_{\theta,t} - g_{\theta,t} - \delta_{\tau,\theta}$, where $g_{\theta,t}$ is the number of sites in $\mathscr{G}_t$ with $\tau_i = \theta$.

The evolution of the process is completely specified by the initial conditions $\mathscr{C}_0$, $\partial\mathscr{C}_0$, and $p_{0,0}(x)$, the above set of equations, and by the deterministic rules used to identify the set $\mathscr{G}_t$. Equations (1)–(4) provide a general mapping of a quenched process, i.e., a deterministic process in a random environment, to an *annealed* stochastic process with a time-dependent randomness. The GPD, Eq. (1), as well as the RTS, depends in general on the whole history of the process. This memory is stored in the distributions of individual variables and generally makes it less likely for an *old* variable to grow than for a *younger* one. The memory effect extends over a time $T = \max_{i,j \in \partial\mathscr{C}_t} \{\tau_i - \tau_j\}$. Indeed if all variables have experienced the same history, no memory effect at all is present and $\mu_{\tau,t} = 1/N_t$.

A mapping between quenched and annealed models was proposed in ref. 10 for the dielectric breakdown model. This was actually in the direction opposite to that discussed here: a stochastic model was mapped to a quenched model. As a result a spatial correlation in the disorder was induced. It is interesting to note that the reverse path discussed here

produces a time correlation, i.e., a memory, in the stochastic model. Moreover, in ref. 10 it was found that this correlation is not present when growth is allowed on all sites (i.e., if $\partial\mathscr{C}_t$ contains all sites that are not in $\mathscr{C}_t$) and results in a disconnected cluster. This is consistent with the evident fact that, in our formalism, no memory is produced in the reverse transformation. This is because the distributions of all sites are conditioned on the same history in disconnected growth. Actually this argument is not rigorous in the case of the quenched version of the dielectric breakdown model since there the random potential is coupled to the Laplace field.[10]

The choice of the initial probability density $p_{0,0}(x)$ is clearly unessential. One can safely restrict attention to the uniform density in $[0, 1]$, since the GPD is invariant under the transformation $x \to \int^x p_{0,0}(y)\, dy$ that maps any density to the uniform one.

The probability densities $p_{\theta,t}(x)$ and $m_t(x \mid \tau)$ are themselves random. Iteration of Eq. (4) implies that

$$p_{\theta,t}(x) = A \prod_{k=t-\theta}^{t-1} M_k(x \mid \tau_k) \tag{5}$$

where A is the normalization constant and τ_k is the τ value of the smallest variable at time k. The noise acts in a multiplicative fashion on the distributions and this may be related to the occurrence of multiscaling in quenched processes.[4]

4. SOLUTION FOR THE EMPIRICAL DISTRIBUTION

The starting point of the analysis of the set of equations (1)–(4) is the relation

$$\sum_{i \in \partial\mathscr{C}_t} \rho_{i,t}(x) + \sum_{i \in \mathscr{C}_t} \rho_{i,t}(x) = |\partial\mathscr{C}_t| + |\mathscr{C}_t| = N_t + \sum_{k=0}^{t-1} (G_k + 1) \tag{6}$$

where $\rho_{i,t}(x)$ is the RTS of the variable ε_i, $G_t + 1 = \sum_\tau g_{\tau,t} + 1 = |\mathscr{G}_t| + 1$ is the total number of sites that grow at time t, and $p_{0,0}(x) = 1$ was used. This identity is, in a sense, a "law of conservation of information." The history of the process up to time t gives only information on how the probability is distributed among the two sums in Eq. (6), but it cannot affect the value of their sum. Equation (6) also introduces the empirical distribution $h_t(x) = \sum_{i \in \partial\mathscr{C}_t} \rho_{i,t}(x)$ that is the histogram of the random variables on the interface (note that the same name in the literature was used for different quantities[9,3]). This is a directly accessible quantity in a computer simulation. Our aim is to derive an analytic expression for $h_t(x)$ under simple and

controllable approximations. It will be shown that $h_t(x)$ provides also information on the other quantities that characterize the growth process, like $\mu_{\tau,t}$ and $n_{\tau,t}$. We start from the case of IP ($\mathscr{G}_t \equiv \varnothing$ and $G_t = 0$), for which our scheme can be compared with rigorous results,[9] and later we will turn to the general case.

4.1. Solution for IP

The first step to derive an equation for the empirical distribution consists in taking the difference of Eq. (6) for $t+1$ and t. Since $\mathscr{C}_{t+1} = \mathscr{C}_t \cup \{i_t\}$ and $\rho_{i_t,t} = m_t(x \mid \tau)$, this is $h_{t+1}(x) = h_t(x) - m_t(x \mid \tau) + N_{t+1} - N_t + 1$. This still contains random quantities so we take the average $\langle \cdot \rangle$ over realizations and we define $\Omega_t = \langle N_t \rangle$ and $\omega_t = \partial_t \Omega_t = \langle N_{t+1} - N_t \rangle$. For a fixed realization of $\mathscr{C}_t$, the average over the realizations $\mathscr{C}_{t+1}$ of $m_t(x \mid \tau)$ yields $\langle m_t(x \mid \tau) \rangle_{\mathscr{C}_t} = \sum_\tau n_{\tau,t} \mu_{\tau,t} m_t(x \mid \tau) = -\partial_x Z_t(x)$. This gives

$$\langle h_{t+1}(x) \rangle = \langle h_t(x) \rangle + \partial_x \langle Z_t(x) \rangle + \omega_t + 1 \tag{7}$$

which is still an exact relation. The *boundary* conditions for this equation are (with the usual convention on uppercase notation)

$$\langle H_t(1) \rangle = \Omega_t \tag{8}$$

$$\langle H_{t+1}(1) \rangle = \langle N_{t+1} \rangle = \Omega_t + \omega_t \tag{9}$$

$$\langle h_{t+1}(0) \rangle = \omega_t + 1 \tag{10}$$

Here the first two are statements on the total number of sites in $\partial \mathscr{C}_t$ and $\partial \mathscr{C}_{t+1}$, while the last one derives from the observation [see Eq. (4)] that $p_{\tau,t}(0) = \delta_{\tau,0}$ for any t and that $\langle h_t(0) \rangle = -\partial_x \langle Z_t(x) \rangle|_{x=0} = n_{0,t}$.

In order to obtain a closed equation for $\langle h_t(x) \rangle$, we have to express both $\langle Z_t(x) \rangle$ and $\langle h_{t+1}(x) \rangle$ as functions of $\langle h_t(x) \rangle$. These are the two main sources of approximations in the solution for the histogram. The second is controllable, since once a solution for $\langle h_t(x) \rangle$ is found, the approximation used for $\langle h_{t+1}(x) \rangle$ can be checked by power expansion in the time variable around t.

The approximation for $\langle Z_t(x) \rangle$ comes from expanding $(1 - P_{\tau,t})^{n_{\tau,t}} = \exp[n_{\tau,t} \log(1 - P_{\tau,t})] \simeq \exp[-n_{\tau,t} P_{\tau,t} + O(P_{\tau,t}^2)]$ in Eq. (2). When x is small the terms of higher order are small compared to the first. On the other hand, when x is close to 1, $Z_t(x)$ is exponentially small in N_t. These considerations justify the approximation of retaining only the first term: $\langle Z_t(x) \rangle \cong \langle \exp[-H_t(x)] \rangle$. The same approximation is obtained in another way with the use of a Poisson transformation[11] in the Appendix.

Note that from Eq. (2), $Z_t(1)=0$, while in the above approximation $\langle Z_t(1)\rangle = \exp(-N_t)$.

Since $h_t(x)$ is a histogram, the number of sites whose variable is found in the xth bin (i.e., $x \leqslant \varepsilon_i < x+dx$) should follow a binomial law with mean $\langle h_t(x)\rangle\, dx$ and variance $\langle h_t(x)\rangle\, dx\,[1-\langle h_t(x)\rangle\, dx/\Omega_t] \cong \langle h_t(x)\rangle\, dx$. This would suggest that $\langle \exp[-H_t(x)]\rangle \approx \exp[-\beta\langle H_t(x)\rangle]$, where $\beta<1$ should account for the first two cumulants, while the central limit theorem should rules out the higher ones. If

$$Z_t(x) \cong \frac{\exp[-\beta\langle H_t(x)\rangle] - \exp[-\beta\Omega_t]}{1-\exp[-\beta\Omega_t]} \tag{11}$$

is put in Eq. (7), we easily find that the conditions (8)–(10) are satisfied only if β is the solution of

$$\beta = 1 - e^{-\beta\Omega_t} \tag{12}$$

The solution $\beta \neq 0$ is exponentially close to 1, which would be consistent with the case in which also the second cumulant of $h_t(x)$ is negligible. From another point of view, $\beta=1$ would result also if the $n_{\tau,t}$ were all independent variables distributed with a Poisson law (see Appendix). We will return later to this point when comparing the results to numerical data. For the moment we take $\langle Z_t(x)\rangle$ as given by Eqs. (11), and (12) so that Eq. (7) becomes

$$\langle h_{t+1}(x)\rangle = \langle h_t(x)\rangle\{1-\exp[-\beta\langle H_t(x)\rangle]\} + \omega_t + 1 \tag{13}$$

We now introduce the function $\phi_t(x) = \langle h_t(x)\rangle/\Omega_t$ and expand $\langle h_{t+1}\rangle$ in the time variable around t: $\langle h_{t+1}(x)\rangle = \Omega_t\phi_t(x) + \omega_t\phi_t(x) + \Omega_t\partial_t\phi_t(x) + \cdots$. Our strategy is to retain only the term proportional to $\phi_t(x)$ and to check on the solution whether the other terms are relevant. Equation (8) implies that $\int_0^1 \phi_t(x) = 1$. The choice $\langle h_{t+1}(x)\rangle = (\Omega_t+\omega_t)\,\phi_t(x)$ satisfies Eqs. (9) and (10), which implies that

$$\phi_t(0) = \frac{\omega_t+1}{\Omega_t+\omega_t} \tag{14}$$

In this approximation, after some algebra, Eq. (13) yields

$$\partial_x\phi_t(x) = \beta\Omega_t\phi_t^2(x)\left[1-\frac{\omega_t}{\omega_t+1}\phi_t(x)\right] \tag{15}$$

which is readily integrated to give $\phi_t(x)$ in implicit form

$$\frac{\omega_t}{\omega_t+1}\log\left(\frac{\Omega_t\phi_t}{\omega_t+1-\omega_t\phi_t}\right) - \frac{1}{\phi_t} = \beta\Omega_t(x-x_c(t)) \tag{16}$$

where

$$x_c(t) = \frac{1}{\beta\Omega_t \phi_t(0)} = \frac{\Omega_t + \omega_t}{\beta\Omega_t(\omega_t + 1)} \tag{17}$$

and Eq. (14) was used for $\phi_t(0)$. It is a matter of algebra to verify that $\int_0^1 \phi_t(x)\,dx = 1$ is satisfied. $\phi_t(x)$ is monotonously increasing [see (15)]. For $x = 0$ its value is of order $1/\Omega_t$, while at $x = x_c(t)$ it attains a finite value $\phi_c \cong 0.7822(1 + \omega_t^{-1})$. The $x_c(t)$ is also very close to the point where the second derivative of ϕ_t vanishes [and $\phi_t = 2(1 + \omega_t^{-1})/3$]. Finally at $x = 1$ it reaches the value $\phi_t(1) = (1 + \omega_t)/[\omega_t + \Omega_t \exp(-\beta\Omega_t)] \cong 1 + \omega_t^{-1}$. Some algebra shows that for $0 < x_c(t) - x \sim 1/\log \Omega_t$ the function ϕ_t is still close to $\phi_t(0)$ and is of the order of $\log \Omega_t/\Omega_t$. It is of the order $\phi_t \sim 1/\log \Omega_t$ for $0 < x_c(t) - x \sim \log\log \Omega_t/\Omega_t$. Above $x_c(t)$ we find that $\phi_t(1) - \phi_t(x) \sim 1/\Omega_t$ for $0 < x - x_c(t) \sim \log \Omega_t/\Omega_t$. So there is a small interval in which $\phi_t(x)$ changes rapidly from $\phi_t(0)$ to $\phi_t(1)$ [in this interval indeed its derivative, Eq. (15), is large].

Equation (16) also allows us to evaluate $\partial_t \phi_t(x)$ and check the validity of the approximation. This can be cast in the form $F(\phi_t, \Omega_t) = 0$, where the dependence on t comes in Ω_t. Then $\partial_t \phi_t = -\omega_t \partial_{\Omega_t} F / \partial_{\phi_t} F$, which reads

$$\partial_t \phi_t(x) = \beta\omega_t \phi_t^2(x) \left[1 - \frac{\omega_t}{\omega_t + 1}\phi_t(x)\right][x - x_c(t)] = \frac{\omega_t}{\Omega_t}(x - x_c)\,\partial_x \phi \tag{18}$$

From the above estimates of ϕ_t we find that $\Omega_t \partial_t \phi_t$ is small in absolute value. At most it has a sharp peak of the order of $\log\log \Omega_t/(\log \Omega_t)^2$ close to x_c. Thus it is vanishingly small w.r.t. $(\Omega_t + \omega_t)\phi_t$ as $\Omega_t \to \infty$, which supports the validity of the approximation. Higher time derivatives of ϕ_t, which can be explicitly calculated in the same way, also yield a negligible contribution as $\Omega_t \to \infty$.

Note that $\Omega_t \partial_t \phi_t$ is of the same order of $\omega_t \phi_t$ up to $x \sim x_c - 1/\log \Omega_t$. The function $\partial_t \phi_t(x)$ decreases below x_c and it increases above x_c; $\partial_t \phi_t$ vanishes very rapidly over a length of the order of $1/\Omega_t$ for $x > x_c$ and $\int_0^1 \partial_t \phi_t(x)\,dx = 0$, apart from exponentially small terms. So the total loss of $\phi_t(x)$ below x_c is compensated by its increase in a small region above x_c.

One unpleasant feature of the approximation used is that $\langle h_t(0)\rangle$ is fixed by the boundary conditions. The undesired consequence of this is that, since $\langle h_t(0)\rangle = \langle N_t - N_{t-1}\rangle = \omega_{t-1}$, Eq. (14) yields a recurrence relation for Ω_t that implies $\lim_{t\to\infty} \omega_t = \infty$. The simplest way to overcome this problem is to take $\langle h_{t+1}(x)\rangle = (\Omega_t + \tilde{\omega}_t)\phi_t(x) + \omega_t - \tilde{\omega}_t$ from the beginning. The only effect of this different choice is to replace ω_t by $\tilde{\omega}_t$ in Eqs. (14)–(18). Using the expansion $\langle h_{t+1}(x)\rangle = \langle h_t(x)\rangle + \partial_t \langle h_t(x)\rangle + \cdots$,

we can control the approximation requiring that the L_2 norm of the error $\Delta\langle h_{t+1}(x)\rangle = \langle h_t(x)\rangle + \partial_t\langle h_t(x)\rangle - [(\Omega_t + \tilde{\omega}_t)\,\phi_t(x) + \omega_t - \tilde{\omega}_t]$ is minimum. This condition fixes the value of $\tilde{\omega}_t$, which, once the calculations are worked out, is

$$\tilde{\omega}_t = \omega_t + \frac{3\omega_t}{2\Omega_t x_c(t)} + O(\Omega_t^{-2}\log\Omega_t) \tag{19}$$

Thus, in the end, this refinement of the approximation implies only a shift of ω_t by an amount of order $1/\Omega_t$.

Figure 1 displays the histogram from 10^6 realizations of $t = 110$ IP clusters compared to the solution $\phi_t(x)$. The values of $\Omega_t \simeq 106$ and $\omega_t \simeq 0.853...$ were computed in the simulation. The accuracy of the solution, as expected, gets worse as x_c is approached.[2] The accuracy improves as Ω_t increases. Note that $\phi_t(x)$ has a sharper character than the numerical histogram, which may be indicative of a too large value of Ω_t or β. The weak point in our scheme, which is also the hardest to deal with, is the approximation on $\langle Z_t(x)\rangle$. The same assumption that $\langle Z_t(x)\rangle$ depends on the RTS through $\langle h_t(x)\rangle$ is by no means obvious.

The histogram $\phi_t(x) = \langle h_t(x)\rangle/\Omega_t$ converges to a step function as $t \to \infty$. This is the behavior that is actually observed for empirical distributions in quenched growth models. In IP the location of the discontinuity coincides with the critical threshold p_c and we find

$$p_c = \lim_{t\to\infty} x_c(t) = \lim_{t\to\infty} \frac{1}{\omega_t + 1} \tag{20}$$

The limit of ω_t as $t \to \infty$ in IP is the surface-to-volume ratio of the cluster. In ref. 9 this relation was derived rigorously for IP. The last relation is very similar to the one that results from a partition function approach for the self-avoiding walk[12] where the critical step fugacity k_c is related to the average number b_t of bonds that can be added to a walk of t steps: $k_c = \lim_{t\to\infty} 1/b_t$. The similarity is even more suggestive in terms of $n_{0,t} = \omega_t + 1$. The relation $p_c = \lim_{t\to\infty} 1/n_{0,t}$ implies that asymptotically the average number of the newly added random variables that are smaller than p_c is one. Figure 2 shows that Eq. (20) is satisfied quite well in IP. The motivation for the choice of the variable $t^{-1/D\nu}$ in Fig. 2 comes from scaling relations in ordinary percolation. The size $\xi(p)$ of the typical cluster is connected with the deviation from p_c as $\xi(p) \sim (p_c - p)^{-\nu}$. This is also related to the mass t via the fractal dimension D: $\xi(p) \sim t^{1/D}$. These scaling relations together imply that $p_c - (1+\omega_t)^{-1} \sim t^{-1/D\nu}$. The slow

[2] The discrepancy for $x > x_c(t)$ is required by the normalization condition.

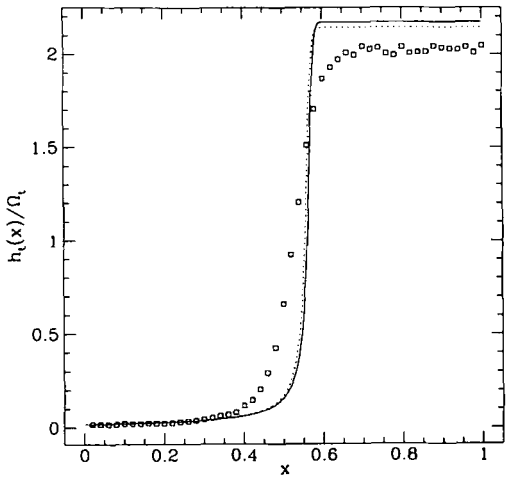

Fig. 1. Empirical distribution $h_t(x)$ evaluated in a computer simulation (□) compared to $\phi_t(x)$. A total of 10^6 clusters of size $t = 110$ were generated. The parameters in $\phi_t(x)$, computed in the simulation, are $\Omega_t \simeq 106$, $\omega_t \simeq 0.853$ (full line), and $\tilde{\omega}_t \simeq 0.875$ (dotted line).

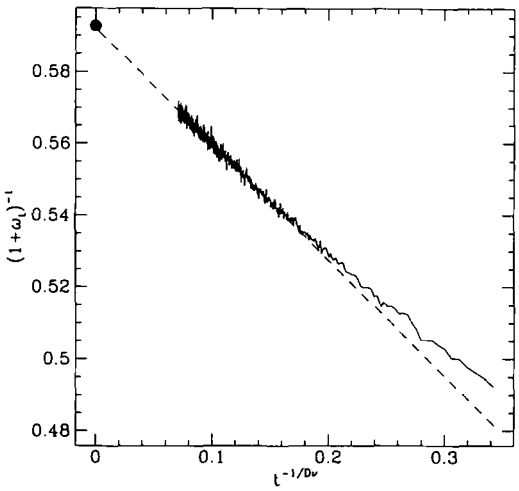

Fig. 2. Effective threshold $p_c(t) = 1/(\omega_t + 1)$ from numerical simulation (10^6 cluster of size $t = 800$) for site invasion percolation on the square lattice. The form $p_c(t) = p_c(\infty) + At^{-1/D\nu}$, suggested by scaling considerations (see text), is used in a linear extrapolation in the range $150 < t < 800$ (dashed line), which yields $p_c(\infty) = 0.5924 \pm 0.0010$, in excellent agreement with estimates of p_c in square site percolation.

convergence to p_c makes it hard to use this method to evaluate the critical threshold: to get $(1+\omega_t)^{-1}$ within 1%, p_c clusters of the size of $t \sim 10^5$ are needed.

Once $\langle h_t(x) \rangle$ is known, we can also evaluate the distribution of the minimum variable in $\partial\mathscr{C}_t$: $\langle m_t(x) \rangle = -\partial_x \langle Z_t(x) \rangle = \omega_t + 1 - \omega_t \phi_t(x)$. In turn, $\langle M_t(x) \rangle = \int_0^x \langle m_t(x) \rangle$ provides an approximate expression for $p_{\tau,t}(x)$ through Eq. (4). Actually we cannot evaluate the average of the product of $M_k(x)$ in Eq. (5), but only the product of the average values $\langle M_t(x) \rangle$. Using Eq. (18) and expanding $\langle M_{t-k}(x) \rangle$ in the time variable around t, we easily find that $\langle M_{t-k}(x) \rangle$ can be replaced by $\langle M_t(x) \rangle$ at the expense of errors of the order of k/Ω_t. Within these approximations the RTS $p_{\tau,t}(x) \simeq A \langle M_t(x) \rangle^\tau$ can be used in Eq. (1) to get an estimate of the GPD:

$$\mu_{\tau,t} = \int_0^1 \frac{M_t^\tau(x)}{\int_x^1 M_t^\tau(y)\, dy} [1 - M_t^\tau(x)]\, dx \tag{21}$$

where a factor $Z_t(x) = 1 - M_t(x)$ has been singled out in the integrand of Eq. (1) and we have suppressed the average symbol. The above integral is easily evaluated numerically.[3] Since the limit $\lim_{t \to \infty} \mu_{\tau,t} = \mu_{\tau,\infty}$ is finite and $\mu_{\tau,t} = \mu_{\tau,\infty} + O(\Omega_t^{-1})$, we leave the discussion of the GPD for a later paragraph, were $\mu_{\tau,\infty}$ is analyzed.

Finally from the estimate of $p_{\tau,t}(x)$ we can also find an approximated expression for $n_{\tau,t}$. This is achieved expanding both sides of

$$\Omega_t \phi_t(x) = \sum_{\tau=0}^{\infty} \frac{n_{\tau,t}}{\int_0^1 M_t^\tau(y)\, dy} M_t^\tau(x) \tag{22}$$

which defines $\langle h_t(x) \rangle$ in powers of x. All the derivatives of ϕ_t and $M_t(x)$, as well as the integral of $M_t^\tau(x)$, can be evaluated explicitly, so that equating the coefficients of x^k on both sides yields a system of linear equations for $n_{\tau,t}$. We just mention here the leading term in $1/\Omega_t$, which is

$$n_{\tau,t} = \omega_t + \frac{1}{\tau+1} + O(\Omega_t^{-1}) \tag{23}$$

This concludes the exposition of the results for the IP model. In summary we have been able to give an estimate of all the quantities appearing in Eqs. (1)–(4). Due to the approximations used, which are too severe with respect to time correlations, our results, especially those concerning the RTS, the GPD, and $n_{\tau,t}$, are not expected to meet more than qualitative agreement with the actual quantities.

[3] The integrand can be expressed as a function of $M_t(x)$ alone and then a change of integration $\dot{M}_t(x)\, dx = dM$ leads to a definite integral in M.

4.2. The General Case

The extension to the case $\mathscr{G}_t \neq \varnothing$ of the procedure outlined for the IP model follows the same guidelines. The only difference is that in taking the difference of Eq. (6) for $t+1$ and t the new term $\sum_{i \in \mathscr{G}_t} \rho_{i,t}(x)$ appears. The origin of this new term is model-dependent. The way in which the sites in $\mathscr{G}_t$ are chosen in $\partial\mathscr{C}_t$ may depend on geometric properties of the interface in a neighborhood of the site with the minimum RV. Examples of this sort will be discussed in the next section. One could also think of a mechanism of growth independent of the location of the smallest RV. Think, for example, of a modification of IP in which the sites in $\mathscr{G}_t$ are chosen randomly with equal probability in $\partial\mathscr{C}_t$. This generalization would include IP and the Eden model, when $G_t = |\mathscr{G}_t| \gg 1$, as particular cases. For this model

$$\left\langle \sum_{i \in \mathscr{G}_t} \rho_{i,t}(x) \right\rangle = \frac{\gamma_t}{\Omega_t} \langle h_t(x) \rangle \tag{24}$$

where $\gamma_t = \langle G_t \rangle$. We will discuss later how to improve this equation to account for the situation in which the growth events of $\mathscr{G}_t$ and of i_t are not independent.

The boundary conditions (8) and (9) still apply, while that on $\langle h_{t+1}(x) \rangle$ must be modified to

$$\langle h_{t+1}(0) \rangle = \gamma_t + \omega_t + 1 \tag{25}$$

The same approximation (11) for $\langle Z_t(x) \rangle$ is used, but now the same algebra shows that Eqs. (8), (9), and (25) imply that β is a solution of

$$\beta = \left(1 - \frac{\gamma_t}{\Omega_t}\right)(1 - e^{-\beta\Omega_t}) \tag{26}$$

and the equation for the histogram finally becomes

$$\langle h_{t+1}(x) \rangle = \left(1 - \frac{\gamma_t}{\Omega_t}\right) \langle h_t(x) \rangle (1 - e^{-\beta \langle H_t(x) \rangle}) + \gamma_t + \omega_t + 1 \tag{27}$$

As before, $\langle h_t(x) \rangle = \Omega_t \phi_t(x)$ and we take $\langle h_{t+1}(x) \rangle = (\Omega_t + \omega_t)\, \phi_t(x)$. Then Eq. (14) generalizes to

$$\phi_t(0) = \frac{\gamma_t + \omega_t + 1}{\Omega_t + \omega_t} \tag{28}$$

while the solution is given by

$$\frac{\gamma_t + \omega_t}{\gamma_t + \omega_t + 1} \log \left[\frac{(\Omega_t - \gamma_t)\, \phi_t}{\gamma_t + \omega_t + 1 - (\gamma_t + \omega_t)\, \phi_t} \right] - \frac{1}{\phi_t} = \beta\Omega_t [x - x_c(t)] \tag{29}$$

with $x_c(t) = [\beta \Omega_t \phi_t(0)]^{-1}$. The same considerations discussed for the IP case are readily translated to this general case. Again we can control the approximation on $\langle h_{t+1}(x) \rangle$ and eventually refine it with a "renormalized" value $\tilde{\omega}_t$ of ω_t. Also, the evaluation of the RTS, of the GPD (which refers only to the site with the smallest RV), and of $n_{\tau,t}$ follows the same guidelines and does not need further discussion.

As a final remark we discuss a refinement on the approximation of Eq. (24) when the G_t sites are chosen in a way that depends on the site i_t. To improve the approximation on $\sum_{i \in \mathscr{G}_t} \rho_{i,t}(x) = \sum_\tau g_{\tau,t} p_{\tau,t}(x)$, we note that Eq. (24) implies that $\gamma_t n_{\tau,t} = \Omega_t g_{\tau,t}$. This means that the probability of a site in $\partial \mathscr{C}_t$ to be in $\mathscr{G}_t$ is the same for all sites (i.e., γ_t/Ω_t) irrespective of their τ value. Often the mechanism to select the sites in $\mathscr{G}_t$ involves only the close neighborhood of i_t. It is then likely that in this case $g_{\tau,t}$ will be larger for τ small than for $\tau \gg 1$. Within our scheme we can enhance in a simple way the importance of the $\tau = 0$ term in the sum on $\mathscr{G}_t$. This is accomplished taking $g_{\tau,t} = \tilde{\gamma}_t n_{\tau,t}/\Omega_t + (\gamma_t - \tilde{\gamma}_t)\, \delta_{\tau,0}$, so that Eq. (24) becomes

$$\left\langle \sum_{i \in \mathscr{G}_t} \rho_{i,t}(x) \right\rangle = \frac{\tilde{\gamma}_t}{\Omega_t} \langle h_t(x) \rangle + \gamma_t - \tilde{\gamma}_t \tag{30}$$

This is easily seen to provide the same solution outlined above but with γ_t replaced by $\tilde{\gamma}_t$. So finally this refinement only implies a shift in the threshold value. The magnitude $\delta\gamma_t = \gamma_t - \tilde{\gamma}_t$ of this shift is a parameter that must be supplied by consideration of the actual mechanism of growth; however, if $g_{0,t}$ is of the same order as $g_{\tau,t}$ for $\tau > 0$, the shift is expected to be $\delta\gamma_t \sim \gamma_t/\Omega_t$, which is vanishingly small as $\Omega_t \to \infty$. This argument can also be applied to the $\tau = 1$ term. This indeed would yield in Eq. (30) a term which is proportional to $\langle M_t(x) \rangle = 1 - \langle Z_t(x) \rangle$, which can also be expressed as a function of $\langle h_t(x) \rangle$. In principle this procedure could be extended further to deal with $\tau = 2, 3, \ldots$. The extension is straightforward, but since it depends on the specific mechanism of growth, it will not be discussed here.

4.3. The Case $\omega_t = 0$

A particular case worthy of mention is that of $\gamma_t > 0$ and $\omega_t = 0$. Models recently introduced for interface growth[4] and for biological evolution[3] are exactly of this type. We briefly mention the former. In $d = 1 + 1$ dimensions, the interface grows along an infinitely long lattice whose transverse size is L. Cylindrical boundary conditions apply in this direction. The surface is pinned by a random force modeled assigning a RV to every site

of the lattice. The second ingredient of the model is surface tension, which tends to minimize the extension of the interface. In the extreme case (infinite surface tension) the length of the interface is fixed to its minimum possible value (i.e., L) by a height difference constraint that forces the interface on one column to be at the same height or one unit above or below the interface on neighboring columns. When the site i_t with the smallest random variable among those on the interface grows, also all the sites that are necessary to recover the single step constraint are forced to grow. Since the number of sites in the interface is always fixed to L, we are exactly in the case $\omega_t = 0$, while γ_t is the average number of other sites that have to grow to recover the constraint. Other models of the same sort may involve a milder mechanism for the minimization of the interface length, allowing for a stochastic readjustment of the interface in the region close to i_t.

The peculiarity of these models is that since $\Omega_t = L$ is fixed, they evolve in a stationary regime of growth. The guess $\langle h_{t+1}(x) \rangle = \Omega_t \phi_\infty = \langle h_t(x) \rangle$ in Eq. (27) is exact for the steady-state empirical distribution [note indeed that $\partial_t \phi_t(x) \propto \omega_t$]. This forces us to change slightly the notation: for L fixed, we define γ_L as the steady-state average value of γ_t. The discontinuity in the empirical distribution appears in this case in the $L \to \infty$ limit: $p_c = \lim_{L \to \infty} (1 + \gamma_L)^{-1}$. In ref. 3 a derivation of the empirical distribution was given. The result coincides essentially with our $\phi_\infty(x)$, but the derivation was based on the assumption that all the variables in $\partial \mathscr{C}_t$ have the same probability density $\phi_\infty(x)$. This is a misleading assumption, in our opinion, since the difference in the probability densities of the variables in $\partial \mathscr{C}_t$ is actually the origin of the nontrivial behavior of these models. Apart from the discussion of the solution, which again follows the same guidelines outlined for IP, we are also in a position to analyze the dynamics of relaxation to the stationary solution $L\phi_\infty(x)$. This is usually characterized by a relaxation time Γ^{-1} that depends on the size of the system through the dynamical exponent z: $\Gamma \sim L^{-z}$. A suggestion in this respect comes from the fact that Eq. (27) is mapped into that for the IP model, Eq. (13), if $\Omega_t = L - \gamma_t$ and $\omega_t = \gamma_t$. At $t = 0$, in both models, $\langle h_0(x) \rangle$ is a constant and the same function, apart from an overall factor, at time t is obtained iterating t times Eqs. (27) and (13). Since in IP the number of iterations needed to get an interface of $\Omega_t = L$ sites is proportional to $\Omega_t/\omega_t = L/\gamma_t$, we expect $\Gamma^{-1} \propto L/\gamma_t$ and $z = 1$.

A more rigorous procedure is to consider Eq. (27) as a functional relation that yields $\phi_{t+1}(x)$ in terms of $\phi_t(x)$: i.e., $\phi_{t+1} = \hat{\mathbf{T}}_{\gamma_t}\{\phi_t\}$. In the following we consider the situation in which the value of γ_t is fixed to $\gamma_t = \gamma_L$. In situations where[4] this parameter is not fixed from the beginning, γ_t reaches its steady-state value with a relaxation law that may be independent of that of the histogram. Here we suppose that γ_t relaxes

faster than the histogram to γ_L or else that it relaxes so slowly that $\phi_t = \phi_\infty^{(\gamma_t)}$ evolves with γ_t adiabatically.[4]

The operator $\hat{\mathbf{T}}_{\gamma_L}$ has the fixed point $\phi_\infty^{(\gamma_L)}$. The dynamical relaxation is related to the damping of a small perturbation in $\phi_t = \phi_\infty^{(\gamma_L)} + \delta\phi$ at the fixed point: $\hat{\mathbf{T}}_{\gamma_L}\{\phi_\infty^{(\gamma_L)} + \delta\phi\} = \phi_\infty^{(\gamma_L)} + e^{-\Gamma}\delta\phi$ to linear order in $\delta\phi$. Actually such a simple relation for the evolution of perturbations does not hold in our case, since

$$\hat{\mathbf{T}}_{\gamma_L}\{\phi_\infty^{(\gamma_L)} + \delta\phi\} = \phi_\infty^{(\gamma_L)} + \delta\phi - \frac{\gamma_L + 1}{L}\,\partial_x \frac{1}{\phi_\infty^{(\gamma_L)}} \int_0^x \delta\phi(x')\,dx' \tag{31}$$

and it is not possible to find $\delta\phi \neq 0$ such that the last term is proportional to $\delta\phi$ itself. A solution is instead possible if we require that

$$\partial_x \frac{1}{\phi_\infty^{(\gamma_L)}} \int_0^x \delta\phi(x')\,dx' = \alpha\,\delta\phi + \eta(1 - \phi_\infty^{(\gamma_L)}) \tag{32}$$

where α and η are parameters. The solution of this equation reads

$$\delta\phi(x) = \frac{\eta}{\gamma_L(1 - \alpha\phi_\infty)^2}\{(\gamma_L x_c - x)\,\partial_x\phi_\infty - \phi[1 + \gamma_L\alpha\phi_\infty(1 - \phi_\infty)]\} \tag{33}$$

The relation

$$\hat{\mathbf{T}}_{\gamma_L}\{\phi_\infty^{(\gamma_L + \delta\gamma_L)}\} = \phi_\infty^{(\gamma_L + \delta\gamma_L)} + (1 - \phi_\infty^{(\gamma_L + \delta\gamma_L)})\,\delta\gamma_L$$

explains the meaning of Eqs. (32) and (33). These state that the operator $\hat{\mathbf{T}}_{\gamma_L}$ cannot be linearized at the fixed point $\phi_\infty^{(\gamma_L)}$, but around the "fixed point" $\phi_\infty^{(\gamma_L + \delta\gamma_L)}$, where $\delta\gamma_L = (1 - \gamma_L/L)(\gamma_L + 1)\eta$. Both $\delta\phi$ and $\delta\gamma_L$ are proportional to η and, as $\eta \propto e^{-\Gamma t} \to 0$, the fixed point $\phi_\infty^{(\gamma_L)}$ is reached. This is a physically sound effect: the steady-state empirical distribution is reached by eliminating the excess of small random variables on the interface. Indeed $\delta\gamma_L > 0$ implies that $\delta x_c(t) < 0$; then $x_c(t) \to x_c(\infty)^-$ as $\delta\gamma_L \to 0$. We finally note that α has to be smaller than $\phi_\infty(1) \cong \gamma_L/(\gamma_L + 1)$ to avoid divergences in (33) for $x \simeq 1$. Taking $\alpha = c\gamma_L/(\gamma_L + 1)$ with $c \leqslant 1$, we get in the end

$$\hat{\mathbf{T}}_{L,\gamma_L}\{\phi_\infty^{(\gamma_L + \delta\gamma_L)} + \delta\phi\} = \phi_\infty^{(\gamma_L + \delta\gamma_L)} + \left(1 - c\,\frac{\gamma_L}{L}\right)\delta\phi \tag{34}$$

which readily yields $\Gamma \simeq \gamma_L/L$ and $z = 1$. Note that the characteristic time diverges as $\gamma_L \to 0$. This limit is achieved in a model of interface growth

[4] The dependence on γ_t is made explicit in ϕ_∞ in this section.

with no surface tension. At each time step the smallest RV is selected among those on top of L adjacent columns and the corresponding column advances one lattice spacing. The process has clearly no steady state, since every column is pinned by always increasing random variables.

5. ASYMPTOTICS

This section is devoted to the analysis of the asymptotic behavior of the quantities appearing in Eqs. (1)–(3) as $\Omega_t \to \infty$. This is the asymptotic time limit when $\omega_t > 0$, while it is the limit $\Omega_t = L \to \infty$ for $\omega_t = 0$. The empirical distribution, as already discussed, tends to a step function with a discontinuity at $p_c = \lim_{t\to\infty} (1+\omega_t+\gamma_t)^{-1}$. The distribution of the minimum instead tends to

$$M_\infty(x) = \min(1, x/p_c) \tag{35}$$

which is also evident from the power expansion of

$$M_t(x) = [1-(\gamma_t+\omega_t)/\Omega_t]\, y + (\gamma_t+\omega_t)\, y^2/(2\Omega_t) + \cdots$$

with $y = x/p_c$. The function $M_\infty(x)$ is the starting point for the calculations in this section. This will be used to evaluate the RTS $p_{\tau,t}(x)$ from Eq. (5) and then the GPD and $n_{\tau,t}$ with a procedure already sketched for the IP case. These quantities will finally allow us to evaluate the "avalanche size" distribution to be defined later. With this introductory outline we wish to make explicit from the beginning the approximations involved in this section. This is actually severe with respect to time correlations since the average of the product of $M_t(x)$ in the evaluation of the RTS, Eq. (5), is substituted by the product of the average. Furthermore, we take $M_{t-k}(x) \cong M_t(x)$ for $k \leqslant \tau$ and

$$p_{\tau,t} = \frac{M_t^\tau(x)}{\int_0^1 M_t^\tau(x')\, dx'} \tag{36}$$

which is a more controllable approximation, since, as discussed in Section 4.1, it only introduces an error of order τ/Ω_t. The results derived within these approximation are, however, expected to give qualitative information on the actual behavior of the models considered.

With this proviso, taking $M_t(x) = M_\infty(x)$ in Eq. (36) and using the resulting RTS in Eq. (1), we find the following expression for the GPD:

$$\mu_{\tau,\infty} = \int_0^1 \frac{(\tau+1)\, y^\tau(1-y)}{1+(\tau+1)(\gamma_t+\omega_t)-y^{\tau+1}}\, dy \tag{37}$$

This is easily evaluated numerically, but a simple closed expression is not available. The fact that $\lim_{\omega_t \to \infty} \mu_{\tau,t} = \mu_{\tau,\infty}$ is finite excludes the possibility of a multifractal behavior of the GPD, which is observed in other growth models that, like IP, produce fractal patterns. Moreover, we note that $\mu_{\tau,\infty} \simeq \tau^{-2}/(\gamma_t + \omega_t) + O(\tau^{-3})$ for $\tau \gg 1$.

We can next investigate the asymptotic distribution of times, that is, $n_{\tau,\infty} = \lim_{t \to \infty} n_{\tau,t}$. This is done by taking the limit on both sides of Eq. (2) and using the above approximation for the RTS. It is easy to realize that, for $x > p_c$, both limits yield zero. Instead for $y = x/p_c < 1$ this reads

$$1 - y = \exp\left\{ \sum_{\tau=0}^{\infty} n_{\tau,\infty} \log\left[\frac{1 + (\tau+1)\,\omega_t - y^{\tau+1}}{1 + (\tau+1)(\gamma_t + \omega_t)} \right] \right\}$$

Differentiation of both sides yields

$$1 = (1-y) \sum_{\tau=0}^{\infty} n_{\tau,\infty} \frac{(\tau+1)\,y^{\tau}}{1 + (\tau+1)(\gamma_t + \omega_t) - y^{\tau+1}}$$

The normalization of the asymptotic GPD $\sum_{\tau=0}^{\infty} n_{\tau,\infty}\mu_{\tau,\infty} = 1$ is indeed recovered, integrating between 0 and 1 this equation. For $y = 0$ we again find $n_{0,\infty} = \gamma_t + \omega_t + 1$. A system of linear equations for $n_{\tau,\infty}$ results from equating the coefficients of y^k in the expansion of the r.h.s. to zero for $k > 0$. A simple expression is not possible for any τ. If $\tau - 1$ is a prime integer, we find

$$n_{\tau,\infty} = [\gamma_t + \omega_t + 1/(\tau+1)][1 + (\gamma_t + \omega_t + 1)^{-\tau}]$$

For general values of τ more terms appear in the second bracket, while the first factor (the leading one for $\tau \to \infty$) remains unchanged. The result $n_{\tau,\infty} = \gamma_t + \omega_t + 1/(1+\tau)$ derives easily from the same procedure if $Z_t(x) \simeq \exp[-\beta H_t(x)]$ instead of Eq. (2) is used. This agrees with the estimate of $n_{\tau,t}$ given in Section 4.1, but it does not yield the normalization of the asymptotic GPD.

The distributions $\mu_{\tau,\infty}$ and $n_{\tau,\infty}$ allow the evaluation of the avalanche size distribution. In a process like IP, growth occurs in avalanche events: when a site becomes part of the cluster it will likely be followed by its nearest neighbors, these by their neighbors, and so on. This happens because a site may "open" a region of the lattice where the random pinning force is weak, so that further growth will occur in this region. In this case the initiator site i_0, which grows at time t_0, is followed in the evolution by sites with $\varepsilon_{i_t} < \varepsilon_{i_0}$ which are causally connected with it since they were not in $\partial\mathscr{C}_{t_0}$ when i_0 was selected. The avalanche process stops at a time $t_0 + s$ when all the newly added perimeter sites have $\varepsilon_i > \varepsilon_{i_0}$ for $i \in \partial\mathscr{C}_{t_0+s}$. Alter-

natively we can say that the avalanche stops when a site that is "older" than i_0 grows. If this happens at time t_0+s, then for this site we must have $\tau \geqslant s$. The probability that the avalanche stops at time t_0+s it then just $\sum_{\tau \geqslant s} n_{\tau,t_0}\mu_{\tau,t_0}$. The probability $W_t(s)$ that the avalanche initiated by the growth event at time t is bigger than (or equal to) s is then the probability that the avalanche did not stop before time $t+s$, that is,

$$W_t(s) = \prod_{k=1}^{s} \left(1 - \sum_{\tau \geqslant k} n_{\tau,t+k}\mu_{\tau,t+k}\right)$$

We are interested in the behavior of this quantity for $t \to \infty$ and $1 \ll s \ll t$. In this limit we can use the above estimates of $n_{\tau,\infty}$ and $\mu_{\tau,\infty}$. The sum is easily evaluated as an integral and some further algebra yields

$$\lim_{t \to \infty} W_t(s) = W_\infty(s) \sim s^{-\kappa}$$

The appearance of the power law crucially depends on the fact that $n_{\tau,\infty}\mu_{\tau,\infty}$ decays exactly as τ^{-2}, while the value of the exponent is given by $\kappa = \lim_{\tau \to \infty} \tau^2 n_{\tau,\infty}\mu_{\tau,\infty} = 1$. This exponent implies that the average avalanche size diverges logarithmically. In real models this divergence has a power-law character and the exponent κ is smaller than 1. For example, in IP the avalanche size distribution is related to the cluster size distribution of the percolation at p_c and $\kappa = 5/91$. The discrepancy is related to the approximations used; it is important to stress, however, that already at this level one is able to capture the correct behavior of $\mu_{\tau,\infty}$ which is responsible for the appearance of a power law in the avalanche size distribution. A more sensible approximation is needed to obtain a more realistic value of the limit that defines κ.

In some situations one is interested on the growth process inside a small region of the interface. This occurs in the application of theoretical schemes like the real-space renormalization group or the fixed scale transformation[7] (FST). The latter was applied successfully to IP[8] yielding very accurate results for the exponents. As shown in ref. 8, the scale-invariant dynamics, which is the base of the FST approach, can be defined using conditional probability and the RTS. Actually this was done in ref. 8 mising one of the passages in Eqs. (1)–(4). More precisely, Eq. (3) was not considered and the RTS of the growing site i_t was used instead of $M_t(x \mid \tau)$ in Eq. (4). In other words, the information that the growing site had the smallest variable was not accounted for properly. A consequence of this is that the distribution function that comes in the evolution can only have the form of a simple power $p_{\tau,t}(x) = (k+1)\,x^k$. These of course cannot reproduce a step function in the large-time limit and would correspond to the case $p_c = 1$.

Even if this procedure seems conceptually wrong, it is justified by the following considerations and it is in the end almost correct. The key point is that in the FST a small region of an infinite or very large cluster is considered. In Eq. (3) for $m_t(x \mid \tau)$ we can factor out $Z_t(x)$,

$$m_t(x \mid \tau) = \frac{p_{\tau,t}(x)}{\mu_{\tau,t}[1 - P_{\tau,t}(x)]} Z_t(x)$$

which may be approximated by $Z_t(x) \simeq \max(1 - x/p_c, 0)$ if the interface is very large. Since this vanishes at p_c, the effect of $Z_t(x)$ is to "renormalize" p_c to 1. Moreover, since $P_{\tau,t}(x)$ is small for $x < p_c$, the approximation in which $m_t(x \mid \tau) \cong p_{\tau,t}(x)$ is a sensible one. This suggests that, in spite of the conceptual error, the precision of the results of the FST approach to IP is not accidental. The above equations also give a hint on how to refine the calculations of the fractal dimension in the FST. Furthermore, the formalism outlined in Section 3 can be used in the investigation of more complex issues, such as the scale invariance of the dynamics or the dynamical renormalization group of quenched growth processes.

6. OUTLOOK

The approach described in Section 3 to quenched growth models provides information that is not available in direct numerical simulation. The knowledge of the GPD allows the evaluation of the probability of a given realization. This opens the way to the exact statistics that may be used, for example, in small-cluster-size studies in connection with finite-size theory. In principle the formalism can be used in computer simulation iterating Eqs. (1)–(4) numerically. However, in practice this would be much more demanding than direct simulation of the quenched model. A simple remedy to this problem consists in a "variational" approach in which the RTS is fitted to a parameter-dependent trial distribution. The numerical task can be reduced to that of finding the evolution of the parameters that optimize the fit. Another example of the use of the GPD is the numerical evaluation of the avalanche size distribution along the lines outlined in Section 5.

Another interesting application of the RTS is in models where the disorder is coupled to an external field as in the quenched version of the dielectric breakdown model[10] or in recently proposed models for fracture propagation.[13] The extension of the formalism to include such effects poses no difficulties in principle. This also opens the way to theoretical investigations on the fractal properties of these models in the spirit of the FST approach.[7] A similar extension is also possible for models for domain

wall dynamics at zero temperature in the random field Ising model,[6] where along with the disorder also the occupation of nearest neighbors has to be considered.

On the analytical side, together with the improvement of the approximations used in this paper, the description outlined in Section 4 can be combined to other schemes like the FST and in the application to specific models at least at a mean-field level.

Research on all these lines is currently active and we hope that this paper will suggest and stimulate other contributions.

APPENDIX. THE POISSON TRANSFORMATION

The average of $Z_t(x)$ on the realizatons $\mathscr{C}_t$ is given by

$$\langle Z_t(x)\rangle = \sum_{\{\mathscr{C}_t\}} \mathscr{W}(\mathscr{C}_t)\, Z_t(x, \mathscr{C}_t)$$

where $\mathscr{W}(\mathscr{C}_t)$ is the probability of a given cluster. The dependence of $Z_t(x)$ on $\mathscr{C}_t$ comes primarily because of the explicit exponents $n_{\tau,t}$ in Eq. (2). Neglecting other dependences, which cause in the functions $P_{\tau,t}(x)$ through $M_t(x \mid \tau)$ [see Eq. (4)], the average can be performed on the probability $\mathscr{P}\{n_{\tau,t}\}$ of a given set $\{n_{\tau,t}\}$ (note that $P_{\tau,t}$ does not depend on the distribution of times $\{n_{\tau,t}\}$ at the same time). The average is easily evaluated using the Poisson transformation[11]

$$\mathscr{P}(\mathscr{C}_{t-1}, \{n_{\tau,t}\}) = \int \mathscr{F}(\mathscr{C}_{t-1}, \{\alpha_\tau\}) \prod_k d\alpha_k \frac{\alpha_k^{n_{k,t}}}{n_{k,t}!} e^{-\alpha_k}$$

where the new variables α_k have of the meaning of the average value of $n_{k,t}$. With this transformation the sum on $n_{k,t}$ can be performed explicitly in evaluating the averages, with the result

$$\langle h_t(x)\rangle_{\mathscr{C}_t} = \sum_\tau \langle \alpha_\tau \rangle\, P_{\tau,t}(x) = N_t \phi(x, t)$$

and

$$\langle Z_t(x)\rangle_{\mathscr{C}_t} = \left\langle \exp\left[-\sum_\tau \alpha_\tau P_{\tau,t}(x) \right] \right\rangle = \left\langle \exp\left[-\int_0^x h_t(y)\, dy \right] \right\rangle$$

In the average also the term $n_{\tau,t} = 0$ for all τ is considered and this yields $\langle Z_t(1)\rangle = e^{-N_t} \neq 0$.

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